

ON THE SIGN CHANGES OF A WEIGHTED DIVISOR PROBLEM

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ABSTRACT. Let $S(x; \frac{a_1}{q_1}, \frac{a_2}{q_2}) = \sum'_{mn \leq x} \cos(2\pi m \frac{a_1}{q_1}) \sin(2\pi n \frac{a_2}{q_2})$ with $x \geq q_1 q_2$, $1 \leq a_i \leq q_i$, and $(a_i, q_i) = 1$ ($i = 1, 2$). We study the sign changes of $S(x; \frac{a_1}{q_1}, \frac{a_2}{q_2})$, and prove that for a sufficiently large constant C , $S(x; \frac{a_1}{q_1}, \frac{a_2}{q_2})$ changes sign in the interval $[T, T + C\sqrt{T}]$ for any large T . Meanwhile, we show that for a small constant c' , there exist infinitely many subintervals of length $c'\sqrt{T} \log^{-7} T$ in $[T, 2T]$ where $\pm S(t; \frac{a_1}{q_1}, \frac{a_2}{q_2}) > c_5(q_1 q_2)^{\frac{3}{4}} t^{\frac{1}{4}}$ always holds.

1. INTRODUCTION

1.1. **Dirichlet divisor problem.** Let $d(n)$ be the Dirichlet divisor function, and $D(x) = \sum_{n \leq x} d(n) = \sum_{n_1 n_2 \leq x} 1$ denote the summatory function. In 1849, Dirichlet proved that

$$D(x) = x \log x + (2\gamma - 1)x + O(\sqrt{x}),$$

where γ is the Euler constant.

Let

$$\Delta(x) = D(x) - x \log x - (2\gamma - 1)x$$

be the error term in the asymptotic formula for $D(x)$. Dirichlet's divisor problem consists of determining the smallest α , for which $\Delta(x) \ll x^{\alpha+\varepsilon}$ holds for any $\varepsilon > 0$. Clearly, Dirichlet's result implies that $\alpha \leq \frac{1}{2}$. Since then, there are many improvements on this estimate. The best to-date is given by Huxley [7, 8], reads

$$(1.1) \quad \Delta(x) \ll x^{\frac{131}{416}} \log^{\frac{26947}{8320}} x.$$

It is widely conjectured that $\alpha = \frac{1}{4}$ is admissible and is the best possible.

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Since $\Delta(x)$ exhibits considerable fluctuations, one natural way to study the upper bounds is to consider the moments.

In 1904, Voronoi [15] showed that

$$\int_1^T \Delta(x) dx = \frac{T}{4} + O(T^{\frac{3}{4}}).$$

Later, in 1922 Cramér [3] proved the mean square formula

$$\int_1^T \Delta(x)^2 dx = cT^{\frac{3}{2}} + O(T^{\frac{5}{4}+\varepsilon}), \quad \forall \varepsilon > 0,$$

where c is a positive constant. In 1983, Ivić [9] used the method of large values to prove that

$$(1.2) \quad \int_1^T |\Delta(x)|^A dx \ll T^{1+\frac{A}{4}+\varepsilon}, \quad \forall \varepsilon > 0$$

for each fixed $0 \leq A \leq \frac{35}{4}$. The range of A can be extended to $\frac{262}{27}$ by the estimate (1.1). In 1992, Tsang [13] obtained the asymptotic formula

$$(1.3) \quad \int_1^T \Delta(x)^k dx = c_k T^{1+\frac{k}{4}} + O(T^{1+\frac{k}{4}-\delta_k}), \quad \text{for } k = 3, 4,$$

with positive constants c_3, c_4 , and $\delta_3 = \frac{1}{14}$, $\delta_4 = \frac{1}{23}$. Ivić and Sargos [10] improved the values δ_3, δ_4 to $\delta'_3 = \frac{7}{20}$, $\delta'_4 = \frac{1}{12}$, respectively. Heath-Brown [5] in 1992 proved that for any integer $k < A$, where A satisfies (1.2), the limit

$$c_k = \lim_{X \rightarrow \infty} X^{-1-\frac{k}{4}} \int_1^X \Delta(x)^k dx$$

exists. Then, there followed a series of investigations on explicit asymptotic formula of the type (1.3) for larger values of k . In 2004, Zhai [16] established asymptotic formulas for $3 \leq k \leq 9$.

At the beginning of the 20th century, Voronoi [15] proved the remarkable exact formula that

$$\Delta(x) = -\frac{2}{\pi} \sqrt{x} \sum_{n=1}^{\infty} \frac{d(n)}{\sqrt{n}} (K_1(4\pi\sqrt{nx})) + \frac{\pi}{2} Y_1(4\pi\sqrt{nx}),$$

where K_1, Y_1 are the Bessel functions, and the series on the right-hand side is boundedly convergent for x lying in each fixed closed interval.

Heath-Brown and Tsang [6] studied the sign changes of $\Delta(x)$. They proved that for a suitable constant $C > 0$, $\Delta(x)$ changes sign on the interval $[T, T + C\sqrt{T}]$ for every sufficiently large T . Here the length \sqrt{T} is almost best possible since they proved that in the interval $[T, 2T]$ there are many subintervals of length $\gg \sqrt{T} \log^{-5} T$ such that $\Delta(x)$ does not change sign in any of these subintervals.

1.2. A weighted divisor problem. Recently, Berndt et al [1, 2] considered a weighted divisor function $\sum'_{mn \leq x} \cos(2\pi m\theta_1) \sin(2\pi n\theta_2)$, and got an analogue of Voronoi's formula as follows.

Let J_1 be the ordinary Bessel function. If $0 < \theta_1, \theta_2 < 1$ and $x > 0$, then

$$\begin{aligned} & \sum'_{mn \leq x} \cos(2\pi m\theta_1) \sin(2\pi n\theta_2) \\ &= -\frac{\cot(\pi\theta_2)}{4} + \frac{\sqrt{x}}{4} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \left\{ \frac{J_1\left(4\pi\sqrt{(m+\theta_1)(n+\theta_2)x}\right)}{\sqrt{(m+\theta_1)(n+\theta_2)}} \right. \\ & \quad + \frac{J_1\left(4\pi\sqrt{(m+1-\theta_1)(n+\theta_2)x}\right)}{\sqrt{(m+1-\theta_1)(n+\theta_2)}} - \frac{J_1\left(4\pi\sqrt{(m+\theta_1)(n+1-\theta_2)x}\right)}{\sqrt{(m+\theta_1)(n+1-\theta_2)}} \\ & \quad \left. - \frac{J_1\left(4\pi\sqrt{(m+1-\theta_1)(n+1-\theta_2)x}\right)}{\sqrt{(m+1-\theta_1)(n+1-\theta_2)}} \right\}. \end{aligned}$$

Denote

$$S\left(x; \frac{a_1}{q_1}, \frac{a_2}{q_2}\right) = \sum'_{mn \leq x} \cos\left(2\pi m \frac{a_1}{q_1}\right) \sin\left(2\pi n \frac{a_2}{q_2}\right).$$

In [11], we got for $x \geq q_1 q_2$, $1 \leq a_i \leq q_i$, $(a_i, q_i) = 1$ ($i = 1, 2$) that

$$S\left(q_1 q_2 x; \frac{a_1}{q_1}, \frac{a_2}{q_2}\right) \ll q_1 q_2 x^{\frac{131}{416}} (\log x)^{\frac{26947}{8320}},$$

$$(1.4) \quad \int_1^T S\left(q_1 q_2 x; \frac{a_1}{q_1}, \frac{a_2}{q_2}\right) dx \ll q_1 q_2 T^{\frac{3}{4}}.$$

If $T \gg (q_1 q_2)^\varepsilon$ is large enough, then for $2 \leq k \leq 9$ we proved

$$(1.5) \quad \int_1^T S^k\left(q_1 q_2 x; \frac{a_1}{q_1}, \frac{a_2}{q_2}\right) dx = (q_1 q_2)^k C_k \int_1^T x^{\frac{k}{4}} dx + o\left((q_1 q_2)^k T^{1+\frac{k}{4}}\right),$$

where C_k are explicit constants.

Here we study $S\left(x; \frac{a_1}{q_1}, \frac{a_2}{q_2}\right)$ further and give some more estimates about it.

NOTATIONS. For a real number t , let $[t]$ be the largest integer no greater than t , $\{t\} = t - [t]$, $\psi(t) = \{t\} - \frac{1}{2}$, $\|t\| = \min(\{t\}, 1 - \{t\})$, $e(t) = e^{2\pi i t}$. \mathbb{C} , \mathbb{R} , \mathbb{Z} , \mathbb{N} denote the set of complex numbers, of real numbers, of integers, and of natural numbers, respectively; $f \asymp g$ means that both $f \ll g$ and $f \gg g$ hold. Throughout this paper, ε denote sufficiently small positive constants, and \mathcal{L} denotes $\log T$.

2. MAIN RESULTS

In this paper, we will discuss the sign changes of $S(x; \frac{a_1}{q_1}, \frac{a_2}{q_2})$ and get the following

Theorem 2.1. *Let $c_1 > 0$ be a sufficiently small constant and $c_2 > 0$ be a sufficiently large constant, $q_1 \geq 2$, $q_2 \geq 3$, $1 \leq a_i \leq q_i$ and $(a_i, q_i) = 1$ ($i = 1, 2$). For any real-valued function $|f(t)| \leq c_1 t^{\frac{1}{4}}$, the function $S(t; \frac{a_1}{q_1}, \frac{a_2}{q_2}) + f(t)$ changes sign at least once in the interval $[T, T + c_2 \sqrt{q_1 q_2 T}]$ for every sufficiently large $T \geq (q_1 q_2)^{1+\varepsilon}$. In particular, there exist $t_1, t_2 \in [T, T + c_2 \sqrt{q_1 q_2 T}]$ such that $S(t_1; \frac{a_1}{q_1}, \frac{a_2}{q_2}) \geq c_1 t_1^{\frac{1}{4}}$ and $S(t_2; \frac{a_1}{q_1}, \frac{a_2}{q_2}) \leq -c_1 t_2^{\frac{1}{4}}$.*

Theorem 2.2. *There exist positive absolute constants c_3, c_4, c_5 such that, for any large parameter $T \geq (q_1 q_2)^{1+\varepsilon}$, there are at least $c_3 \sqrt{T} \log^7 T$ disjoint subintervals of length $c_4 \sqrt{T} \log^{-7} T$ in $[T, 2T]$, such that $\pm S(t; \frac{a_1}{q_1}, \frac{a_2}{q_2}) > c_5 (q_1 q_2)^{\frac{3}{4}} t^{\frac{1}{4}}$, whenever t lies in any of these subintervals. Moreover, we have the estimate*

$$\text{meas}\{t \in [T, 2T] : \pm S(t; \frac{a_1}{q_1}, \frac{a_2}{q_2}) > c_5 (q_1 q_2)^{\frac{3}{4}} t^{\frac{1}{4}}\} \gg T.$$

We also study the Ω -result of the error term in the asymptotic formula (1.5) for odd k by using Theorem 2.2. Define

$$\mathcal{F}_k(q_1 q_2 x; \frac{a_1}{q_1}, \frac{a_2}{q_2}) := (q_1 q_2)^{-k} \int_1^T S^k(q_1 q_2 x; \frac{a_1}{q_1}, \frac{a_2}{q_2}) dx - C_k T^{1+\frac{k}{4}}.$$

We have the following

Theorem 2.3. *The estimate*

$$\mathcal{F}_k(q_1 q_2 T; \frac{a_1}{q_1}, \frac{a_2}{q_2}) = \Omega(T^{\frac{1}{2}+\frac{k}{4}} \mathcal{L}^{-7})$$

holds for any fixed odd integer $k \geq 3$ and every sufficiently large $T \geq (q_1 q_2)^\varepsilon$.

Remark 2.1. *Although at the present moment we can only prove (1.5) for $2 \leq k \leq 9$, Theorem 2.3 holds for any odd $k \geq 2$.*

Remark 2.2. *We can get the same or similar conclusions with all presented here for $\sum'_{mn \leq x} \sin(2\pi n \frac{a_1}{q_1}) \sin(2\pi m \frac{a_2}{q_2})$ and $\sum'_{mn \leq x} \cos(2\pi n \frac{a_1}{q_1}) \cos(2\pi m \frac{a_2}{q_2})$ with the same approach.*

3. THE VORONOI-TYPE FORMULA FOR $S(x; \frac{a_1}{q_1}, \frac{a_2}{q_2})$

In [11], we proved an analogue of Voronoi's formula for $S(q_1 q_2 x; \frac{a_1}{q_1}, \frac{a_2}{q_2})$.

Denote

$$\Delta d_2(n; a_1, q_1, a_2, q_2) = d(n; a_1, q_1, a_2, q_2) + d(n; -a_1, q_1, a_2, q_2) \\ - d(n; a_1, q_1, -a_2, q_2) - d(n; -a_1, q_1, -a_2, q_2),$$

$$\Delta d_{2,1}(n, H, J; a_1, q_1, a_2, q_2) \\ = \sum'_{\substack{n=hl \\ 1 \leq h \leq H \\ h \leq l \leq 2^{J+1}h \\ h \equiv a_2 \pmod{q_2} \\ l \equiv a_1 \pmod{q_1}}} 1 + \sum'_{\substack{n=hl \\ 1 \leq h \leq H \\ h \leq l \leq 2^{J+1}h \\ h \equiv a_2 \pmod{q_2} \\ l \equiv -a_1 \pmod{q_1}}} 1 - \sum'_{\substack{n=hl \\ 1 \leq h \leq H \\ h \leq l \leq 2^{J+1}h \\ h \equiv -a_2 \pmod{q_2} \\ l \equiv a_1 \pmod{q_1}}} 1 - \sum'_{\substack{n=hl \\ 1 \leq h \leq H \\ h \leq l \leq 2^{J+1}h \\ h \equiv -a_2 \pmod{q_2} \\ l \equiv -a_1 \pmod{q_1}}} 1,$$

$$\Delta d_{2,2}(n, H, J; a_1, q_1, a_2, q_2) \\ = \sum'_{\substack{n=hl \\ 1 \leq h \leq H \\ h \leq l \leq 2^{J+1}h \\ h \equiv a_2 \pmod{q_2} \\ l \equiv a_1 \pmod{q_1}}} 1 + \sum'_{\substack{n=hl \\ 1 \leq h \leq H \\ h \leq l \leq 2^{J+1}h \\ h \equiv a_2 \pmod{q_2} \\ l \equiv -a_1 \pmod{q_1}}} 1 - \sum'_{\substack{n=hl \\ 1 \leq h \leq H \\ h \leq l \leq 2^{J+1}h \\ h \equiv -a_2 \pmod{q_2} \\ l \equiv a_1 \pmod{q_1}}} 1 - \sum'_{\substack{n=hl \\ 1 \leq h \leq H \\ h \leq l \leq 2^{J+1}h \\ h \equiv -a_2 \pmod{q_2} \\ l \equiv -a_1 \pmod{q_1}}} 1.$$

Let $J = \lceil \frac{\mathcal{L} + 2 \log q_1 q_2 - 4 \log \mathcal{L}}{\log 2} \rceil$, $H \geq 2$ be a parameter to be determined, and $T^\varepsilon < y \leq \min(H^2, (q_1 q_2)^2 T) \mathcal{L}^{-4}$. Suppose $T \leq x \leq 2T$. Then

$$(3.1) \quad S(q_1 q_2 x; \frac{a_1}{q_1}, \frac{a_2}{q_2}) = R_0(x; y) + R_{12}(x; y, H) + R_{21}(x; y, H) \\ + G_{12}(x; H) + G_{21}(x; H) + O(q_1 q_2 \mathcal{L}^3),$$

where

$$(3.2) \quad R_0(x; y) = \frac{q_1 q_2 x^{\frac{1}{4}}}{4\sqrt{2}\pi} \sum_{n \leq y} \frac{\cos(4\pi\sqrt{nx} - \frac{3\pi}{4})}{n^{\frac{3}{4}}} \Delta d_2(n; a_1, q_1, a_2, q_2), \\ R_{12}(x; y, H) = \frac{q_1 q_2 x^{\frac{1}{4}}}{4\sqrt{2}\pi} \sum_{y < n \leq 2^{J+1}H^2} \frac{\cos(4\pi\sqrt{nx} - \frac{3\pi}{4})}{n^{\frac{3}{4}}} \Delta d_{2,1}(n, H, J; a_1, q_1, a_2, q_2), \\ R_{21}(x; y, H) = \frac{q_1 q_2 x^{\frac{1}{4}}}{4\sqrt{2}\pi} \sum_{y < n \leq 2^{J+1}H^2} \frac{\cos(4\pi\sqrt{nx} - \frac{3\pi}{4})}{n^{\frac{3}{4}}} \Delta d_{2,2}(n, H, J; a_1, q_1, a_2, q_2), \\ G_{12}(x; H) = O\left(q_2 \sum_{n_1 \leq q_1 \sqrt{T}} \min\left(1, \frac{1}{H \left\| \frac{q_1 x}{n_1} - \frac{r_2}{q_2} \right\|}\right)\right), \\ G_{21}(x; H) = O\left(q_1 \sum_{n_2 \leq q_2 \sqrt{T}} \min\left(1, \frac{1}{H \left\| \frac{q_2 x}{n_2} - \frac{r_1}{q_1} \right\|}\right)\right).$$

4. PROOF OF THEOREM 2.1

In this section, we prove Theorem 2.1 following the approach of [6].

Let n_0 be the smallest integer n , such that $\Delta d_2(n; a_1, q_1, a_2, q_2) \neq 0$. By the definition of $\Delta d_2(n; a_1, q_1, a_2, q_2)$, it is easy to see that $\Delta d_2(n_0; a_1, q_1, a_2, q_2) = 1$ or -1 , and $n_0 = \min\{a_1, q_1 - a_1\} \times \min\{a_2, q_2 - a_2\}$, which suggests $n_0 < \frac{1}{4}q_1q_2$.

Suppose $|f(t)| \leq c_1 t^{\frac{1}{4}}$. Let

$$S^*(t) = 4\sqrt{2}\pi(q_1q_2)^{-1}t^{-\frac{1}{2}}\left(S(q_1q_2t^2; \frac{a_1}{q_1}, \frac{a_2}{q_2}) + f(q_1q_2t^2)\right), \quad \text{for } t \geq 1.$$

Define

$$K_\zeta(u) := (1 - |u|)(1 + \zeta \sin(4\pi\alpha\sqrt{n_0}u)) \quad \text{for } |u| \leq 1,$$

with $\zeta = 1$ or -1 , and $\alpha > n_0^{\frac{1}{2}}$ a large number.

Set $\zeta' = -\Delta d_2(n_0; a_1, q_1, a_2, q_2)\zeta$. Then it is easy to see that $\zeta' = 1$ or -1 , and Theorem 2.1 follows from Lemma 4.1 below.

Lemma 4.1. *Suppose $T \gg (q_1q_2)^\varepsilon$ is a large parameter. Then for each $\sqrt{T} \leq t \leq \sqrt{2T}$, we have*

$$\int_{-1}^1 S^*(t + \alpha u) K_\zeta(u) du = \frac{\zeta'}{2n_0^{\frac{3}{4}}} \sin(4\pi t\sqrt{n_0} - \frac{3}{4}\pi) + O(\alpha^{-2} + t^{-\frac{1}{2}}\mathcal{L}^3 + c_1(q_1q_2)^{-\frac{3}{4}}).$$

Proof. Let $J = \lceil \frac{\mathcal{L} + 2\log q_1q_2 - 4\log \mathcal{L}}{\log 2} \rceil$, $H \geq 2$ be a parameter to be determined, and $T^\varepsilon < y \leq \min(H^2, (q_1q_2)^2T)\mathcal{L}^{-4}$. From (3.1), we have

$$(4.1) \quad S^*(t) = R_0^*(t; y) + R_{12}^*(t; y, H) + R_{21}^*(t; y, H) + 4\sqrt{2}\pi(q_1q_2)^{-1}t^{-\frac{1}{2}}f(q_1q_2t^2) \\ + O(t^{-\frac{1}{2}}(G_{12}^*(t; H) + G_{21}^*(t; H))) + O(t^{-\frac{1}{2}}\mathcal{L}^3),$$

where

$$R_0^*(t; y) = \sum_{n \leq y} \frac{\cos(4\pi t\sqrt{n} - \frac{3\pi}{4})}{n^{\frac{3}{4}}} \Delta d_2(n; a_1, q_1, a_2, q_2), \\ R_{12}^*(t; y, H) = \sum_{y < n \leq 2^{J+1}H^2} \frac{\cos(4\pi t\sqrt{n} - \frac{3\pi}{4})}{n^{\frac{3}{4}}} \Delta d_{2,1}(n, H, J; a_1, q_1, a_2, q_2), \\ R_{21}^*(t; y, H) = \sum_{y < n \leq 2^{J+1}H^2} \frac{\cos(4\pi t\sqrt{n} - \frac{3\pi}{4})}{n^{\frac{3}{4}}} \Delta d_{2,2}(n, H, J; a_1, q_1, a_2, q_2), \\ G_{12}^*(t; H) = O\left(\frac{1}{q_1} \sum_{n_1 \leq q_1\sqrt{T}} \min\left(1, \frac{1}{H\left\|\frac{q_1t^2}{n_1} - \frac{r_2}{q_2}\right\|}\right)\right), \\ G_{21}^*(t; H) = O\left(\frac{1}{q_2} \sum_{n_2 \leq q_2\sqrt{T}} \min\left(1, \frac{1}{H\left\|\frac{q_2t^2}{n_2} - \frac{r_1}{q_1}\right\|}\right)\right).$$

Denote

$$R^*(t) = R_0^*(t; y) + R_{12}^*(t; y, H) + R_{21}^*(t; y, H), \quad G^*(t) = G_{12}^*(t; H) + G_{21}^*(t; H).$$

Then

$$(4.2) \quad S^*(t) = R^*(t) + 4\sqrt{2}\pi(q_1 q_2)^{-1} t^{-\frac{1}{2}} f(q_1 q_2 t^2) + O(t^{-\frac{1}{2}} G^*(t)) + O(t^{-\frac{1}{2}} \mathcal{L}^3).$$

We first consider $\int_{-1}^1 G^*(t + \alpha u) du$. Noting that

$$\min\left(1, \frac{1}{H\|r\|}\right) = \sum_{h=-\infty}^{\infty} a(h) e(hr)$$

with

$$a(0) \ll H^{-1} \log H, \quad a(h) \ll \min\left(H^{-1} \log H, h^{-2} H\right), \quad h \neq 0.$$

We have

$$\begin{aligned} & \int_{-1}^1 G_{12}^*(t + \alpha u; H) du \\ &= \frac{1}{q_1} \sum_{h=-\infty}^{\infty} a(h) \sum_{n_1 \leq q_1 \sqrt{T}} e\left(\frac{hq_1 t^2}{n_1} - \frac{hr_2}{q_2}\right) \int_{-1}^1 e\left(\frac{2hq_1 t \alpha u + hq_1 \alpha^2 u^2}{n_1}\right) du \\ &\ll |a(0)| \sqrt{T} + \frac{1}{q_1} \sum_{h=1}^{\infty} |a(h)| \sum_{n_1 \leq q_1 \sqrt{T}} \frac{n_1}{hq_1 t \alpha} \\ &\ll H^{-1} T^{\frac{1}{2}} \log H + \sum_{h=1}^H H^{-1} \log H T (ht \alpha)^{-1} + \sum_{h=H}^{\infty} HT (t \alpha)^{-1} h^{-3} \\ &\ll H^{-1} T^{\frac{1}{2}} \log^2 H, \end{aligned}$$

where the first derivative test was used. This estimate remain valid with G_{12}^* replaced by G_{21}^* , which yields

$$(4.3) \quad \int_{-1}^1 G^*(t + \alpha u) du \ll H^{-1} T^{\frac{1}{2}} \log^2 H.$$

Now we estimate $\int_{-1}^1 R^*(t + \alpha u) K_{\zeta}(u) du$. By the elementary formula

$$\begin{aligned} & \cos\left(4\pi(t + \alpha u)\sqrt{n} - \frac{3\pi}{4}\right) \\ &= \cos\left(4\pi t\sqrt{n} - \frac{3\pi}{4}\right) \cos(4\pi \alpha u \sqrt{n}) - \sin\left(4\pi t\sqrt{n} - \frac{3\pi}{4}\right) \sin(4\pi \alpha u \sqrt{n}), \end{aligned}$$

we get

$$\int_{-1}^1 \cos\left(4\pi(t + \alpha u)\sqrt{n} - \frac{3\pi}{4}\right) (1 - |u|) (1 + \zeta \sin(4\pi \alpha \sqrt{n_0} u)) du = I_1 + I_2,$$

with

$$\begin{aligned}
I_1 &= \cos\left(4\pi t\sqrt{n} - \frac{3\pi}{4}\right) \int_{-1}^1 \cos(4\pi\alpha u\sqrt{n})(1-|u|)(1+\zeta\sin(4\pi\alpha\sqrt{n_0}u))du \\
&= \cos\left(4\pi t\sqrt{n} - \frac{3\pi}{4}\right) \int_{-1}^1 \cos(4\pi\alpha u\sqrt{n})(1-|u|)du, \\
I_2 &= \sin\left(4\pi t\sqrt{n} - \frac{3\pi}{4}\right) \int_{-1}^1 \sin(4\pi\alpha u\sqrt{n})(1-|u|)(1+\zeta\sin(4\pi\alpha\sqrt{n_0}u))du \\
&= \zeta \sin\left(4\pi t\sqrt{n} - \frac{3\pi}{4}\right) \int_{-1}^1 \sin(4\pi\alpha u\sqrt{n})(1-|u|)\sin(4\pi\alpha\sqrt{n_0}u)du \\
&= \frac{\zeta}{2} \sin\left(4\pi t\sqrt{n} - \frac{3\pi}{4}\right) \int_{-1}^1 (1-|u|)\cos(4\pi\alpha u(\sqrt{n}-\sqrt{n_0}))du \\
&\quad - \frac{\zeta}{2} \sin\left(4\pi t\sqrt{n} - \frac{3\pi}{4}\right) \int_{-1}^1 (1-|u|)\cos(4\pi\alpha u(\sqrt{n}+\sqrt{n_0}))du.
\end{aligned}$$

By using

$$\int_0^1 (1-u)\cos(Au)du \ll |A|^{-2} \quad A \neq 0,$$

we have

$$I_1 \ll \alpha^{-2}n^{-1}, \quad I_2 = \begin{cases} \frac{\zeta}{2} \sin\left(4\pi t\sqrt{n_0} - \frac{3\pi}{4}\right) + O(\alpha^{-2}n_0^{-1}), & n = n_0, \\ O(\alpha^{-2}(\sqrt{n}-\sqrt{n_0})^{-2}), & n \neq n_0, \end{cases}$$

which suggests

$$\begin{aligned}
&\int_{-1}^1 \cos\left(4\pi(t+\alpha u)\sqrt{n} - \frac{3\pi}{4}\right) K_\zeta(u)du \\
&= \begin{cases} \frac{\zeta}{2} \sin\left(4\pi t\sqrt{n_0} - \frac{3\pi}{4}\right) + O(\alpha^{-2}n_0^{-1}), & n = n_0, \\ O(\alpha^{-2}(\sqrt{n}-\sqrt{n_0})^{-2}), & n \neq n_0. \end{cases}
\end{aligned}$$

Take $H = T$, $y = T^{\frac{1}{2}}$. Then clearly $n_0 < y$. Thus we get

$$\begin{aligned}
(4.4) \quad &\int_{-1}^1 R^*(t+\alpha u)K_\zeta(u)du \\
&= -\frac{\zeta}{2n_0^{\frac{3}{4}}} \sin\left(4\pi t\sqrt{n_0} - \frac{3\pi}{4}\right) \Delta d_2(n_0; a_1, q_1, a_2, q_2) + O\left(\sum_{n>n_0} \frac{\alpha^{-2}n^{-\frac{3}{4}}d(n)}{(\sqrt{n}-\sqrt{n_0})^2}\right) \\
&= -\frac{\zeta}{2n_0^{\frac{3}{4}}} \sin\left(4\pi t\sqrt{n_0} - \frac{3\pi}{4}\right) \Delta d_2(n_0; a_1, q_1, a_2, q_2) + O(\alpha^{-2}),
\end{aligned}$$

by using $\sum_{n>n_0} \frac{d(n)}{n^{\frac{3}{4}}(\sqrt{n}-\sqrt{n_0})^2} \ll 1$.

Note that $H = T$, $t \asymp T^{\frac{1}{2}}$. From (4.2)-(4.4), we see

$$\begin{aligned}
& \int_{-1}^1 S^*(t + \alpha u) K_{\zeta}(u) du \\
&= -\frac{\zeta}{2n_0^{\frac{3}{4}}} \sin\left(4\pi t\sqrt{n_0} - \frac{3\pi}{4}\right) \Delta d_2(n_0; a_1, q_1, a_2, q_2) + O(\alpha^{-2}) \\
&\quad + O\left((q_1 q_2)^{-1} t^{-\frac{1}{2}} \sup_{|u| \leq 1} f(q_1 q_2 (t + \alpha u)^2)\right) + O\left(t^{-\frac{1}{2}} H^{-1} T^{\frac{1}{2}} \mathcal{L}^2\right) + O\left(t^{-\frac{1}{2}} \mathcal{L}^3\right) \\
&= \frac{\zeta'}{2n_0^{\frac{3}{4}}} \sin\left(4\pi t\sqrt{n_0} - \frac{3\pi}{4}\right) + O(\alpha^{-2}) + O(c_1 (q_1 q_2)^{-\frac{3}{4}}) + O(t^{-\frac{1}{2}} \mathcal{L}^3).
\end{aligned}$$

Thus we complete the proof of Lemma 4.1 \square

5. THE MEAN VALUE OF $S(q_1 q_2 x; \frac{a_1}{q_1}, \frac{a_2}{q_2})$ IN SHORT INTERVALS

Suppose $T \gg (q_1 q_2)^{\varepsilon}$ is a large parameter, $1 \leq h \leq \frac{1}{2}\sqrt{T}$. Denote $S(q_1 q_2 x) = S(q_1 q_2 x; \frac{a_1}{q_1}, \frac{a_2}{q_2})$. In this section we shall estimate the integral

$$I(T, h) = \int_1^T (S(q_1 q_2(x+h)) - S(q_1 q_2 x))^2 dx,$$

which would play an important role in the proof of Theorem 2.2. This type of integral was studied for the error term in the mean square of $\zeta(\frac{1}{2} + it)$ by Good [4], for the error term in the Dirichlet divisor problem by Jutila [12] and for the error term in Weyl's law for Heisenberg manifold by Tsang and Zhai [14]. Here we follow the approach of Tsang and Zhai [14] and prove the following

Lemma 5.1. *The estimate*

$$I(T, h) \ll (q_1 q_2)^2 h T \log^3 \frac{\sqrt{T}}{h} + (q_1 q_2)^2 T \mathcal{L}^6$$

holds uniformly for $1 \leq h \leq \frac{1}{2}\sqrt{T}$.

Proof. Write

$$(5.1) \quad I(T, h) = \int_1 + \int_2,$$

where

$$\begin{aligned}
\int_1 &= \int_1^{100 \max(h^2, T^{\frac{2}{3}})} (S(q_1 q_2(x+h)) - S(q_1 q_2 x))^2 dx, \\
\int_2 &= \int_{100 \max(h^2, T^{\frac{2}{3}})}^T (S(q_1 q_2(x+h)) - S(q_1 q_2 x))^2 dx.
\end{aligned}$$

From (1.5), we see that

$$(5.2) \quad \int_1 \ll (q_1 q_2)^2 (h^3 + T) \ll (q_1 q_2)^2 T h.$$

For \int_2 , first we estimate the integral

$$(5.3) \quad J(U, h) = \int_U^{2U} (S(q_1 q_2(x+h)) - S(q_1 q_2 x))^2 dx, \quad 100 \max(h^2, T^{\frac{2}{3}}) \leq U \leq T.$$

Let $T = 2U$ in (3.1). Then

$$\begin{aligned} S(q_1 q_2 x) &= R_0(x; y) + R_{12}(x; y, H) + R_{21}(x; y, H) \\ &\quad + G_{12}(x; H) + G_{21}(x; H) + O(q_1 q_2 \log^3 U). \end{aligned}$$

Take $H = U$, $y = \min(\frac{1}{2}U h^{-1}, U \log^{-6} U)$. From [11, Lemma 6.2 and Lemma 6.5], we see

$$\begin{aligned} \int_U^{2U} |G_{12}(x; H) + G_{21}(x; H)|^2 dx &\ll (q_1 q_2)^2 U \log U, \\ \int_U^{2U} |R_{12}(x; y, H) + R_{21}(x; y, H)|^2 dx &\ll (q_1 q_2)^2 U^{\frac{3}{2}} y^{-\frac{1}{2}} \log^3 U. \end{aligned}$$

Thus we get

$$\begin{aligned} (5.4) \quad \int_U^{2U} (S(q_1 q_2 x) - R_0(x; y))^2 dx &\ll (q_1 q_2)^2 U^{\frac{3}{2}} y^{-\frac{1}{2}} \log^3 U + (q_1 q_2)^2 U \log^6 U \\ &\ll (q_1 q_2)^2 U h^{\frac{1}{2}} \log^3 U + (q_1 q_2)^2 U \log^6 U. \end{aligned}$$

We now estimate the integral $\int_U^{2U} (R_0(x+h; y) - R_0(x; y))^2 dx$. From (3.2), we have

$$(5.5) \quad R_0(x+h; y) - R_0(x; y) = F_1(x) + F_2(x),$$

where

$$\begin{aligned} F_1(x) &= \frac{q_1 q_2}{4\sqrt{2}\pi} ((x+h)^{\frac{1}{4}} - x^{\frac{1}{4}}) \sum_{n \leq y} \frac{\Delta d_2(n; a_1, q_1, a_2, q_2)}{n^{\frac{3}{4}}} \cos\left(4\pi \sqrt{n(x+h)} - \frac{3\pi}{4}\right), \\ F_2(x) &= \frac{q_1 q_2 x^{\frac{1}{4}}}{4\sqrt{2}\pi} \sum_{n \leq y} \frac{\Delta d_2(n; a_1, q_1, a_2, q_2)}{n^{\frac{3}{4}}} \left(\cos\left(4\pi \sqrt{n(x+h)} - \frac{3\pi}{4}\right) - \cos\left(4\pi \sqrt{nx} - \frac{3\pi}{4}\right)\right). \end{aligned}$$

From [11, Lemma 6.3], we get

$$(5.6) \quad \int_U^{2U} F_1^2(x) dx \ll h^2 U^{-2} \int_U^{2U} R_0^2(x+h) dx \ll (q_1 q_2)^2 h^2 U^{-\frac{1}{2}}.$$

For the mean square of $F_2(x)$, we see

$$(5.7) \quad F_2^2 = F_{21} + F_{22},$$

with

$$\begin{aligned}
F_{21}(x) &= \frac{(q_1 q_2)^2}{32\pi^2} x^{\frac{1}{2}} \sum_{n \leq y} \frac{\Delta d_2^2(n; a_1, q_1, a_2, q_2)}{n^{\frac{3}{2}}} \\
&\quad \times \left(\cos \left(4\pi \sqrt{n(x+h)} - \frac{3\pi}{4} \right) - \cos \left(4\pi \sqrt{nx} - \frac{3\pi}{4} \right) \right)^2, \\
F_{22}(x) &= \frac{(q_1 q_2)^2}{32\pi^2} x^{\frac{1}{2}} \sum_{\substack{m, n \leq y \\ m \neq n}} \frac{\Delta d_2(m; a_1, q_1, a_2, q_2) \Delta d_2(n; a_1, q_1, a_2, q_2)}{(mn)^{\frac{3}{4}}} \\
&\quad \times \left(\cos \left(4\pi \sqrt{m(x+h)} - \frac{3\pi}{4} \right) - \cos \left(4\pi \sqrt{mx} - \frac{3\pi}{4} \right) \right) \\
&\quad \times \left(\cos \left(4\pi \sqrt{n(x+h)} - \frac{3\pi}{4} \right) - \cos \left(4\pi \sqrt{nx} - \frac{3\pi}{4} \right) \right).
\end{aligned}$$

By writing

$$\cos \left(4\pi \sqrt{n(x+h)} - \frac{3\pi}{4} \right) - \cos \left(4\pi \sqrt{nx} - \frac{3\pi}{4} \right) = \sum_{j=0}^1 (-1)^{j+1} \cos \left(4\pi \sqrt{n(x+jh)} - \frac{3\pi}{4} \right),$$

we get

(5.8)

$$\begin{aligned}
F_{22}(x) &= \frac{(q_1 q_2)^2}{32\pi^2} x^{\frac{1}{2}} \sum_{j_1=0}^1 \sum_{j_2=0}^1 (-1)^{j_1+j_2} \sum_{\substack{m, n \leq y \\ m \neq n}} \frac{\Delta d_2(m; a_1, q_1, a_2, q_2) \Delta d_2(n; a_1, q_1, a_2, q_2)}{(mn)^{\frac{3}{4}}} \\
&\quad \times \cos \left(4\pi \sqrt{m(x+j_1h)} - \frac{3\pi}{4} \right) \cos \left(4\pi \sqrt{n(x+j_2h)} - \frac{3\pi}{4} \right) \\
&= : F_{221}(x) + F_{222}(x),
\end{aligned}$$

where

$$\begin{aligned}
F_{221}(x) &= \frac{(q_1 q_2)^2}{64\pi^2} x^{\frac{1}{2}} \sum_{j_1=0}^1 \sum_{j_2=0}^1 (-1)^{j_1+j_2} \sum_{\substack{m, n \leq y \\ m \neq n}} \frac{1}{(mn)^{\frac{3}{4}}} \Delta d_2(m; a_1, q_1, a_2, q_2) \\
&\quad \times \Delta d_2(n; a_1, q_1, a_2, q_2) \cos \left(4\pi \sqrt{m(x+j_1h)} - 4\pi \sqrt{n(x+j_2h)} \right), \\
F_{222}(x) &= \frac{(q_1 q_2)^2}{64\pi^2} x^{\frac{1}{2}} \sum_{j_1=0}^1 \sum_{j_2=0}^1 (-1)^{j_1+j_2+1} \sum_{\substack{m, n \leq y \\ m \neq n}} \frac{1}{(mn)^{\frac{3}{4}}} \Delta d_2(m; a_1, q_1, a_2, q_2) \\
&\quad \times \Delta d_2(n; a_1, q_1, a_2, q_2) \sin \left(4\pi \sqrt{m(x+j_1h)} + 4\pi \sqrt{n(x+j_2h)} \right).
\end{aligned}$$

Let

$$g_{\pm}(x) = 4\pi \sqrt{m(x+j_1h)} \pm 4\pi \sqrt{n(x+j_2h)}.$$

Using

$$(1+t)^{\frac{1}{2}} = 1 + \sum_{v=1}^{\infty} d_v t^v \quad (|t| \leq \frac{1}{2}),$$

with $|d_v| < 1$, we see

$$g_{\pm}(x) = 4\pi\sqrt{x}(\sqrt{m} \pm \sqrt{n}) + 4\pi \sum_{v=1}^{\infty} \frac{d_v h^v}{x^{v-\frac{1}{2}}} (\sqrt{m} j_1^v \pm \sqrt{n} j_2^v).$$

Noting that $m, n \leq y \leq \frac{1}{2}Uh^{-1}$, we have

$$|g'_{\pm}(x)| \gg \frac{1}{\sqrt{x}} |\sqrt{m} \pm \sqrt{n}| \quad (m \neq n).$$

Then by the the first derivative test we get

$$\begin{aligned} \int_U^{2U} F_{221}(x) dx &\ll (q_1 q_2)^2 U \sum_{\substack{m, n \leq y \\ m \neq n}} \frac{\Delta d_2(m; a_1, q_1, a_2, q_2) \Delta d_2(n; a_1, q_1, a_2, q_2)}{(mn)^{\frac{3}{4}} |\sqrt{m} - \sqrt{n}|}, \\ \int_U^{2U} F_{222}(x) dx &\ll (q_1 q_2)^2 U \sum_{\substack{m, n \leq y \\ m \neq n}} \frac{\Delta d_2(m; a_1, q_1, a_2, q_2) \Delta d_2(n; a_1, q_1, a_2, q_2)}{(mn)^{\frac{3}{4}} |\sqrt{m} + \sqrt{n}|}. \end{aligned}$$

From (5.8), we obtain

$$\begin{aligned} (5.9) \quad \int_U^{2U} F_{22}(x) dx &\ll (q_1 q_2)^2 U \sum_{\substack{m, n \leq y \\ m \neq n}} \frac{\Delta d_2(m; a_1, q_1, a_2, q_2) \Delta d_2(n; a_1, q_1, a_2, q_2)}{(mn)^{\frac{3}{4}} |\sqrt{m} - \sqrt{n}|} \\ &\ll (q_1 q_2)^2 U \log^4 y, \end{aligned}$$

where we used the estimate $\sum_{n \leq N} d(n) \ll N \log N$.

By the elementary formulas

$$\cos u - \cos v = -2 \sin\left(\frac{u+v}{2}\right) \sin\left(\frac{u-v}{2}\right), \quad \text{and} \quad \sin^2 u = \frac{1}{2}(1 - \cos 2u),$$

we have

$$\begin{aligned} (5.10) \quad &\int_U^{2U} F_{21}(x) dx \\ &= \frac{(q_1 q_2)^2}{8\pi^2} \sum_{n \leq y} \frac{\Delta d_2^2(n; a_1, q_1, a_2, q_2)}{n^{\frac{3}{2}}} \int_u^{2U} x^{\frac{1}{2}} \\ &\quad \times \sin^2\left(2\pi\sqrt{n(x+h)} + 2\pi\sqrt{nx} - \frac{3\pi}{4}\right) \sin^2\left(2\pi\sqrt{n(x+h)} - 2\pi\sqrt{nx}\right) dx \\ &=: I_{211} + I_{212}, \end{aligned}$$

where

$$I_{211} = \frac{(q_1 q_2)^2}{16\pi^2} \sum_{n \leq y} \frac{\Delta d_2^2(n; a_1, q_1, a_2, q_2)}{n^{\frac{3}{2}}} \int_U^{2U} x^{\frac{1}{2}} \sin^2(2\pi\sqrt{n(x+h)} - 2\pi\sqrt{nx}) dx,$$

$$I_{212} = \frac{(q_1 q_2)^2}{16\pi^2} \sum_{n \leq y} \frac{\Delta d_2^2(n; a_1, q_1, a_2, q_2)}{n^{\frac{3}{2}}} \int_U^{2U} x^{\frac{1}{2}} \sin(4\pi\sqrt{n(x+h)} + 4\pi\sqrt{nx})$$

$$\times \sin^2(2\pi\sqrt{n(x+h)} - 2\pi\sqrt{nx}) dx.$$

By the first derivative test, we have

$$L_n(t) := \int_U^t x^{\frac{1}{2}} \sin(4\pi\sqrt{n(x+h)} + 4\pi\sqrt{nx}) dx \ll U n^{-\frac{1}{2}}, \quad U \leq t \leq 2U.$$

Using the integration by parts, we obtain

$$\begin{aligned} & \int_U^{2U} x^{\frac{1}{2}} \sin(4\pi\sqrt{n(x+h)} + 4\pi\sqrt{nx}) \sin^2(2\pi\sqrt{n(x+h)} - 2\pi\sqrt{nx}) dx \\ &= \int_U^{2U} \sin^2(2\pi\sqrt{n(x+h)} - 2\pi\sqrt{nx}) dL_n(x) \\ &= L_n(2U) \sin^2(2\pi\sqrt{n(2U+h)} - 2\pi\sqrt{2nU}) - 2 \int_U^{2U} L_n(x) \left(\frac{\pi\sqrt{n}}{\sqrt{x+h}} - \frac{\pi\sqrt{n}}{\sqrt{x}} \right) \\ & \quad \times \sin(2\pi\sqrt{n(x+h)} - 2\pi\sqrt{nx}) \cos(2\pi\sqrt{n(x+h)} - 2\pi\sqrt{nx}) dx \\ &\ll U n^{-\frac{1}{2}} + U^{\frac{1}{2}} h, \end{aligned}$$

which yields

$$(5.11) \quad \begin{aligned} I_{212} &\ll (q_1 q_2)^2 \sum_{n \leq y} \frac{d^2(n)}{n^{\frac{3}{2}}} (U n^{-\frac{1}{2}} + U^{\frac{1}{2}} h) \\ &\ll (q_1 q_2)^2 (U + U^{\frac{1}{2}} h) \ll (q_1 q_2)^2 U. \end{aligned}$$

By using

$$\sqrt{x+h} = x^{\frac{1}{2}} + hx^{-\frac{1}{2}} + O(h^2 x^{-\frac{3}{2}}), \quad x \geq 100h^2,$$

we get

$$\begin{aligned} \sin^2(2\pi\sqrt{n(x+h)} - 2\pi\sqrt{nx}) &= \sin^2(\pi h n^{\frac{1}{2}} x^{-\frac{1}{2}} + O(h^2 n^{\frac{1}{2}} x^{-\frac{3}{2}})) \\ &= \sin^2(\pi h n^{\frac{1}{2}} x^{-\frac{1}{2}}) + O(h^2 n^{\frac{1}{2}} x^{-\frac{3}{2}}). \end{aligned}$$

Noting that

$$\begin{aligned} \int_U^{2U} x^{\frac{1}{2}} \sin^2(\pi h n^{\frac{1}{2}} x^{-\frac{1}{2}}) dx &\ll \int_U^{2U} x^{\frac{1}{2}} \min(1, h^2 n x^{-1}) dx \\ &\ll \begin{cases} U^{\frac{1}{2}} h^2 n, & n \leq U h^{-2}, \\ U^{\frac{3}{2}}, & n > U h^{-2}, \end{cases} \end{aligned}$$

we have

$$\begin{aligned} (5.12) \quad I_{211} &\ll (q_1 q_2)^2 \sum_{n \leq y} \frac{d^2(n)}{n^{\frac{3}{2}}} \int_U^{2U} x^{\frac{1}{2}} (\sin^2(\pi h n^{\frac{1}{2}} x^{-\frac{1}{2}}) + O(h^2 n^{\frac{1}{2}} x^{-\frac{3}{2}})) dx \\ &\ll (q_1 q_2)^2 \sum_{n \leq y} \frac{d^2(n)}{n^{\frac{3}{2}}} \int_U^{2U} x^{\frac{1}{2}} \sin^2(\pi h n^{\frac{1}{2}} x^{-\frac{1}{2}}) dx + O\left((q_1 q_2 h)^2 \sum_{n \leq y} \frac{d^2(n)}{n}\right) \\ &\ll (q_1 q_2 h)^2 U^{\frac{1}{2}} \sum_{n \leq U h^{-2}} \frac{d^2(n)}{n^{\frac{1}{2}}} + (q_1 q_2)^2 U^{\frac{3}{2}} \sum_{n > U h^{-2}} \frac{d^2(n)}{n^{\frac{3}{2}}} + O((q_1 q_2 h)^2 \log^4 y) \\ &\ll (q_1 q_2)^2 U h \log^3 \frac{\sqrt{U}}{h}, \end{aligned}$$

where we used the well-known estimate $\sum_{n \leq N} d^2(n) \ll N \log^3 N$.

From (5.10)-(5.12), we get

$$(5.13) \quad \int_U^{2U} F_{21}(x) dx \ll (q_1 q_2)^2 U h \log^3 \frac{\sqrt{U}}{h}.$$

Combining (5.7), (5.9) and (5.13), we obtain

$$\int_U^{2U} F_2^2(x) dx \ll (q_1 q_2)^2 U h \log^3 \frac{\sqrt{U}}{h} + (q_1 q_2)^2 U \log^4 y,$$

which together with (5.5), (5.6) yields

$$(5.14) \quad \int_U^{2U} (R_0(x+h; y) - R_0(x; y))^2 dx \ll (q_1 q_2)^2 U h \log^3 \frac{\sqrt{U}}{h} + (q_1 q_2)^2 U \log^4 y.$$

From (5.3), (5.4), and (5.14), it follows that

$$J(U, h) \ll (q_1 q_2)^2 U h \log^3 \frac{\sqrt{U}}{h} + (q_1 q_2)^2 U \log^6 y,$$

which implies

$$(5.15) \quad \int_2 \ll (q_1 q_2)^2 T h \log^3 \frac{\sqrt{T}}{h} + (q_1 q_2)^2 T \mathcal{L}^6,$$

via a splitting argument. Then Lemma 5.1 follows from (5.1), (5.2) and (5.15). \square

6. PROOF OF THEOREM 2.2

In this section, we will give a proof of Theorem 2.2 by following the approach of [14]. We still write $S(q_1 q_2 x) = S(q_1 q_2 x; \frac{a_1}{q_1}, \frac{a_2}{q_2})$. Define

$$S_+(t) = \frac{1}{2}(|S(t)| + S(t)), \quad S_-(t) = \frac{1}{2}(|S(t)| - S(t)).$$

We need the following two lemmas.

Lemma 6.1.

$$\int_T^{2T} S_{\pm}^2(q_1 q_2 t) dt \gg (q_1 q_2)^2 T^{\frac{3}{2}}.$$

Proof. From (1.5) with $k = 2, 4$, by Hölder's inequality, we get

$$\begin{aligned} (q_1 q_2)^2 T^{\frac{3}{2}} &\ll \int_T^{2T} S^2(q_1 q_2 t) dt \ll \left(\int_T^{2T} |S(q_1 q_2 t)| dt \right)^{\frac{2}{3}} \left(\int_T^{2T} S^4(q_1 q_2 t) dt \right)^{\frac{1}{3}} \\ &\ll \left(\int_T^{2T} |S(q_1 q_2 t)| dt \right)^{\frac{2}{3}} (q_1 q_2)^{\frac{4}{3}} T^{\frac{2}{3}}, \end{aligned}$$

which yields

$$(6.1) \quad \int_T^{2T} |S(q_1 q_2 t)| dt \gg q_1 q_2 T^{\frac{5}{4}}.$$

From (1.4), we see

$$\int_T^{2T} S(q_1 q_2 t) dt \ll q_1 q_2 T^{\frac{3}{4}}.$$

Thus, from the definition of $S_{\pm}(q_1 q_2 t)$, we have

$$\int_T^{2T} S_{\pm}(q_1 q_2 t) dt \gg q_1 q_2 T^{\frac{5}{4}}.$$

Then by Cauchy-Schwarz's inequality, we get

$$q_1 q_2 T^{\frac{5}{4}} \ll \left(\int_T^{2T} dt \right)^{\frac{1}{2}} \left(\int_T^{2T} S_{\pm}^2(q_1 q_2 t) dt \right)^{\frac{1}{2}} \ll T^{\frac{1}{2}} \left(\int_T^{2T} S_{\pm}^2(q_1 q_2 t) dt \right)^{\frac{1}{2}},$$

which immediately implies Lemma 6.1. \square

Lemma 6.2. Suppose $2 \leq H_0 \leq \sqrt{T}$. Then

$$\int_T^{2T} \max_{h \leq H_0} (S_{\pm}(q_1 q_2(t+h)) - S_{\pm}(q_1 q_2 t))^2 dt \ll (q_1 q_2)^2 H_0 T \mathcal{L}^7.$$

Proof. Since

$$|S_{\pm}(q_1 q_2(t+h)) - S_{\pm}(q_1 q_2 t)| \leq |S(q_1 q_2(t+h)) - S(q_1 q_2 t)|,$$

it is sufficient to prove that

$$I = \int_T^{2T} \max_{h \leq H_0} (S(q_1 q_2(t+h)) - S(q_1 q_2 t))^2 dt \ll (q_1 q_2)^2 H_0 T \mathcal{L}^7.$$

Write $H_0 = 2^\lambda b$, such that $\lambda \in \mathbb{N}$ and $1 \leq b < 2$. By using Lemma 5.1, we get

$$\begin{aligned} I &\ll \lambda \sum_{\mu \leq \lambda} \sum_{0 \leq \nu \leq 2^\mu} \int_{T+\nu 2^{\lambda-\mu} b}^{2T+\nu 2^{\lambda-\mu} b} (S(q_1 q_2(t+2^{\lambda-\mu} b)) - S(q_1 q_2 t))^2 dt + (q_1 q_2)^2 T \mathcal{L}^2 \\ &\ll \lambda \sum_{\mu \leq \lambda} \sum_{0 \leq \nu \leq 2^\mu} ((q_1 q_2)^2 2^{\lambda-\mu} b T \mathcal{L}^3 + (q_1 q_2)^2 T \mathcal{L}^6) \\ &\ll \lambda \sum_{\mu \leq \lambda} ((q_1 q_2)^2 2^\lambda b T \mathcal{L}^3 + (q_1 q_2)^2 2^\mu T \mathcal{L}^6) \\ &\ll \lambda^2 (q_1 q_2)^2 H_0 T \mathcal{L}^3 + \lambda (q_1 q_2)^2 H_0 T \mathcal{L}^6 \\ &\ll (q_1 q_2)^2 H_0 T \mathcal{L}^7, \end{aligned}$$

where we used the well-known estimate

$$\sum_{x < n \leq x+y} d(n) \ll y \log x, \quad x^\varepsilon < y < x.$$

□

Now we finish the proof of Theorem 2.2. For any function $P(t)$ and $Q(t)$ such that

$$\omega(t) = P^2(t) - 4 \max_{h \leq H_0} (P(t+h) - P(t))^2 - Q^2(t) > 0,$$

we see that $P(t+h)$ has the same sign as $P(t)$, and $|P(t+h)| > \frac{1}{2}|Q(t)|$ for any $0 \leq h \leq H_0$. Take $P(t) = S_\pm(q_1 q_2 t)$ and $Q(t) = \delta q_1 q_2 t^{\frac{1}{4}}$ for a sufficiently small $\delta > 0$. By Lemma 6.1 and Lemma 6.2, we get

$$(6.2) \quad \int_T^{2T} \omega(t) dt \gg (q_1 q_2)^2 T^{\frac{3}{2}} - O\left((q_1 q_2)^2 (H_0 T \mathcal{L}^7 + \delta^2 T^{\frac{3}{2}})\right) \gg (q_1 q_2)^2 T^{\frac{3}{2}},$$

by taking $H_0 = \delta T^{\frac{1}{2}} \mathcal{L}^{-7}$. Let

$$\mathcal{S} = \{t \in [T, 2T] : \omega(t) > 0\}.$$

From (1.5) and (6.2), using Cauchy-Schwarz's inequality, we have

$$\begin{aligned} (q_1 q_2)^2 T^{\frac{3}{2}} &\ll \int_T^{2T} \omega(t) dt \leq \int_{\mathcal{S}} \omega(t) dt \leq \int_{\mathcal{S}} S_\pm^2(q_1 q_2 t) dt \\ &\leq |\mathcal{S}|^{\frac{1}{2}} \left(\int_T^{2T} S^4(q_1 q_2 t) dt \right)^{\frac{1}{2}} \ll |\mathcal{S}|^{\frac{1}{2}} (q_1 q_2)^2 T, \end{aligned}$$

which implies

$$|\mathcal{S}| \gg T.$$

Thus the proof of Theorem 2.2 is completed. \square

7. PROOF OF THEOREM 2.3

Suppose $k \geq 3$ is a fixed odd integer and $T \geq (q_1 q_2)^\varepsilon$ is a large parameter. Set

$$\delta = \begin{cases} -1, & \text{if } C_k \geq 0, \\ 1, & \text{if } C_k < 0, \end{cases}$$

where C_k is defined in (1.5).

By Theorem 2.2, there exists $t \in [T, 2T]$ such that $\delta S(q_1 q_2 u; \frac{a_1}{q_1}, \frac{a_2}{q_2}) > c_5 q_1 q_2 t^{\frac{1}{4}}$ for any $u \in [t, t + H_0]$, with $H_0 = c_4 \sqrt{T} \mathcal{L}^{-7}$. Thus

$$\begin{aligned} c_5^k H_0 t^{\frac{k}{4}} &< (q_1 q_2)^{-k} \int_t^{t+H_0} \delta^k S^k(q_1 q_2 u; \frac{a_1}{q_1}, \frac{a_2}{q_2}) du \\ &= \delta^k C_k \left((t+H_0)^{1+\frac{k}{4}} - t^{1+\frac{k}{4}} \right) + \delta^k \left(\mathcal{F}_k(q_1 q_2(t+H_0); \frac{a_1}{q_1}, \frac{a_2}{q_2}) - \mathcal{F}_k(q_1 q_2 t; \frac{a_1}{q_1}, \frac{a_2}{q_2}) \right) \\ &= \delta^k C_k \left(1 + \frac{k}{4} \right) t^{\frac{k}{4}} H_0 + O(H_0^2 t^{\frac{k}{4}-1}) + \delta^k \left(\mathcal{F}_k(q_1 q_2(t+H_0); \frac{a_1}{q_1}, \frac{a_2}{q_2}) - \mathcal{F}_k(q_1 q_2 t; \frac{a_1}{q_1}, \frac{a_2}{q_2}) \right), \end{aligned}$$

which yields

$$\delta^k \left(\mathcal{F}_k(q_1 q_2(t+H_0); \frac{a_1}{q_1}, \frac{a_2}{q_2}) - \mathcal{F}_k(q_1 q_2 t; \frac{a_1}{q_1}, \frac{a_2}{q_2}) \right) > C_k^* H_0 t^{\frac{k}{4}} (1 + O(H_0 T^{-1})),$$

with

$$C_k^* = c_5^k - \delta^k C_k \left(1 + \frac{k}{4} \right) > 0.$$

Thus we get

$$\left| \mathcal{F}_k(q_1 q_2(t+H_0); \frac{a_1}{q_1}, \frac{a_2}{q_2}) - \mathcal{F}_k(q_1 q_2 t; \frac{a_1}{q_1}, \frac{a_2}{q_2}) \right| \gg H_0 T^{\frac{k}{4}},$$

which immediatly implies Theorem 2.3. \square

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